

The Symplectic-Orthogonal Penner Models

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Abstract

The generating function for the orbifold Euler characteristic of the moduli space of real algebraic curves of genus $2g$ (locally orientable surfaces) with n marked points $\chi^r(\mathfrak{M}_{2g,n})$, is identified with a simple formula. It is shown that the free energy in the continuum limit of both the symplectic and the orthogonal Penner models are almost identical, with the structure $F^{SP/SO}(\mu) = \frac{1}{2}F(\mu) \mp F^{NO}(\mu)$, where $F(\mu)$ is the Penner free energy and $F^{NO}(\mu)$ is the free energy contributions from the non-orientable surfaces. Both of these models have the same critical point as the Penner model.

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1 INTRODUCTION

The formula for the orbifold Euler characteristic of the moduli space of real algebraic curves with n marked points (punctures) was first obtained by Chekhov and Zabrodin, using the orthogonal matrix model technique (skew orthogonal polynomials)[1]. Goulden, Harer and Jackson [2] obtained the same formula, as a special case of a more general formula determined from the parametrized polynomial $\xi_g^n(\gamma)$, when $\gamma = 1/2$, where g is the genus of the algebraic curves and n is the number of punctures. These authors used real symmetric matrix integrals rather than hermitian matrix integrals which are known to give the orbifold Euler characteristic of the moduli space of complex algebraic curves (orientable surfaces) [3, 4]. It was shown in [2] that the real symmetric matrix integral gives rise to two different orbifold Euler characteristics, namely: When g is odd, it is the ordinary orbifold Euler characteristic of the moduli space of complex algebraic curves with n marked points. However, when g is even they obtained the orbifold Euler characteristic of the moduli space of real algebraic curves with n marked points. This contributions is due to the non-orientable surfaces. This model is called the orthogonal Penner model. Mulase and Waldron [5] generalized the Penner model to the symplectic Penner model [5] and found that the orthogonal Penner model almost coincides with the symplectic Penner model, the difference being in that the matrix size of the former is doubled and an overall sign difference appears in the non-orientable surfaces contributions. This was also observed for a general potential $V(x)$ by Chekhov and Eynard [6]¹. Our goals in this paper are two fold: First, We identify the generating function for the orbifold Euler characteristic of the moduli space of real algebraic curves when g is even which turns out to be a simple formula, this is done in section 2. The second goal is to obtain the continuum limit of the symplectic-orthogonal Penner models. This will be carried out in section 3. Like the ordinary Penner model [7, 8], we show that the free energy of the symplectic-orthogonal Penner models in the continuum limit are related to the orbifold Euler characteristic of the moduli space without punctures. Both have the same critical point $t = 1$. This is the same critical point of the ordinary Penner model [8].

2 The Generating Function of the Penner Symplectic-Orthogonal Matrix Models

The Penner model for hermitian matrix integral provides an effective tool to compute the orbifold Euler characteristic of the moduli space of smooth algebraic curves defined over \mathbb{C} with an arbitrary number of marked points [4]. While the asymptotic expansion of the Penner model for real symmetric and quaternionic matrix integral is the generating function of the orbifold Euler characteristic of the moduli space of real algebraic curves [2].

The asymptotic expansion of the free energy of the Penner model is constructed from the logarithm of Gaussian integral of self-adjoint matrices. The Gaussian integral of self-adjoint matrices as a

¹The formula relating the symplectic and the orthogonal matrix models for a general potential $V(x)$ reads:

$$-\log \int |\Delta(\lambda)|^{2\alpha} e^{-N\sqrt{\alpha} \sum_{i=1}^N V(\lambda_i)} = \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} N^{2-2g-k} \left(\sqrt{\alpha} - \sqrt{\alpha^{-1}} \right)^k F_{g,k},$$

where $F_{g,k}$ are the corresponding correction to the free energy. Here $\alpha = 1/2$ for the orthogonal matrices and $\alpha = 2$ for the symplectic ones.

function of the coupling constants $t = (t_1, t_2, t_3, \dots)$, and the size of the matrix N , is given by

$$Z^{(2\alpha)}(t, N) = \frac{\int [dX]_{(2\alpha)} \exp\left(-\frac{1}{4}\text{tr } X^2 + \sum_{j=1}^{\infty} \frac{t_j}{2j} \text{tr } X^j\right)}{\int [dX]_{(2\alpha)} \exp\left(-\frac{1}{4}\text{tr } X^2\right)}, \quad (1)$$

where the matrix variable X is constructed from $N \times N$ real symmetric and complex anti symmetric matrices. Using the Penner substitution given by

$$t_j = -(\sqrt{t})^{j-2}, \quad j = 3, 4, \dots, 2m \quad \text{with} \quad t_1 = t_2 = 0, \quad (2)$$

where \sqrt{t} is defined for $\text{Re}(t) > 0$, then the free energy of the Penner model reads

$$F(t, N, \alpha) = \lim_{m \rightarrow \infty} \log \left(\frac{\int [dX]_{(2\alpha)} \exp\left(-\sum_{j=2}^{2m} \frac{t^{j/2-1}}{j} \text{tr } X^j\right)}{\int [dX]_{(2\alpha)} \exp\left(-\frac{1}{2}\text{tr } X^2\right)} \right), \quad (3)$$

where α takes the values $\frac{1}{2}, 1$ or 2 depending on whether we study orthogonal, unitary or symplectic ensembles respectively. Also note that the sum over j in the above equation contains the quadratic term i.e $j = 2$, as well as the interaction terms starting from $j \geq 3$ with conventional normalization². This integral is computable since the matrix variable X is diagonalizable, $X \rightarrow \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ where λ 's are the eigenvalues of the matrix X , therefore the integral becomes

$$F(t, N, \alpha) = \lim_{m \rightarrow \infty} \log \left(\frac{\int_{\mathbb{R}^N} \Delta^{2\alpha}(\lambda) \prod_{i=1}^N \exp\left(-\sum_{j=2}^{2m} \frac{t^{j/2-1}}{j} \lambda_i^j\right) d\lambda_i}{\int_{\mathbb{R}^N} \Delta^{2\alpha}(\lambda) \prod_{i=1}^N \exp\left(-\frac{\lambda_i^2}{2}\right) d\lambda_i} \right), \quad (4)$$

where $\Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$, is the Vandermonde determinant.

Using the Selberg integration formula, Stirling's formula for $\Gamma(1/t)$ and the asymptotic analysis of [9], the above integral can be evaluated for integer values of α , and the result is

$$\begin{aligned} F(t, N, \alpha) = & \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} N t^{2m-1} \\ & + \sum_{m=1}^{\infty} \sum_{i=0}^{N-1} \sum_{j=1}^{\alpha} (-1)^{m-1} \frac{1}{m} (N-1-i)(i\alpha+j)^m t^m, \end{aligned} \quad (5)$$

where B_{2n} are the Bernoulli numbers. The free energy of the original Penner model (integral over hermitian matrices) is obtained by setting $\alpha = 1$ in (5)

$$F(t, N, 1) = \sum_{\substack{g \geq 0, n > 0 \\ 2-2g-n < 0}} \frac{(2g+n-3)!(2g-1)}{(2g)!n!} B_{2g} N^n (-t)^{2g+n-2}, \quad (6)$$

identifying n with the number of faces f_{Γ} , and g as the genus of the triangulated Riemann surfaces. The free energy $F(t, N, 1)$ is considered the generating function of the orbifold Euler characteristic $\chi^c(\mathfrak{M}_{g,n})$ of the moduli space of Riemann surfaces of genus g and n punctures given by

$$\chi^c(\mathfrak{M}_{g,n}) = (-1)^n \frac{(2g+n-3)!(2g-1)}{(2g)!n!} B_{2g}. \quad (7)$$

² To obtain a conventional normalization, we rescale $X \rightarrow 2^{1/2} X$ in equation (1) and absorb all but one power of 2 in the couplings t .

The partition function of the Penner model can be evaluated using the orthogonal polynomial technique [3], given by

$$\left(\frac{\sqrt{2\pi t}(et)^{-(t^{-1})}}{\Gamma(\frac{1}{t})} \right)^N \prod_{p=1}^N (1+pt)^{(N-p)}. \quad (8)$$

By setting $\alpha = 2$ in equation (5) one may show that the free energy of the symplectic Penner model [5] reads:

$$\begin{aligned} F(t, N, 2) = & \frac{1}{2} \sum_{\substack{g \geq 0, n > 0 \\ 2-2g-n < 0}} \frac{(2g+n-3)!(2g-1)}{(2g)!n!} B_{2g} (2N)^n (-t)^{2g+n-2} \\ & - \frac{1}{2} \sum_{\substack{q \geq 0, n > 0 \\ 1-2q-n < 0}} \frac{(2q+n-2)!(2^{2q-1}-1)}{(2q)!n!} B_{2q} (2N)^n (-t)^{2q+n-1}. \end{aligned} \quad (9)$$

As we can see from the above equation, the first term is half the free energy of the Penner model with the size of the matrix doubled i.e. $\frac{1}{2}F(t, 2N, 1)$. This term comes from the contributions of the orientable surfaces, while the second term is the contributions from the non-orientable surfaces of even genus $g = 2q$ with n marked points, where the term

$$\chi^r(\mathfrak{M}_{2q,n}) := (-1)^n \frac{1}{2} \frac{(2q+n-2)!(2^{2q-1}-1)}{(2q)!n!} B_{2q}, \quad (10)$$

is the orbifold Euler characteristic of the moduli space of smooth real algebraic curves of genus $g = 2q$ with n marked points [2]. As a consequence, the free energy of the symplectic Penner model is the generating function of two different orbifold Euler characteristics namely $\chi^c(\mathfrak{M}_{2q,n})$ and $\chi^r(\mathfrak{M}_{g,n})$. The first one is generated by

$$\log \left[\left(\frac{\sqrt{2\pi t}(et)^{-(t^{-1})}}{\Gamma(\frac{1}{t})} \right)^{2N} \prod_{p=1}^{2N} (1+pt)^{(2N-p)} \right]. \quad (11)$$

One of our objectives in this paper is to obtain the generating function for $\chi^r(\mathfrak{M}_{2q,n})$. In so doing, we begin by rewriting the asymptotic expansion of the symplectic Penner model given in equation (5) as two separate generating functions in which the first is given by equation (11) and the second one is identified with the generating function of the orbifold Euler characteristic of smooth real algebraic curves of genus $g = 2q$ with n marked points $\chi^r(\mathfrak{M}_{2q,n})$. From equation (5), the symplectic Penner model is

$$F(t, N, 2) = \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} N t^{2m-1} + \sum_{m=1}^{\infty} \sum_{i=0}^{N-1} \sum_{j=1}^2 (-1)^{m-1} \frac{1}{m} (N-1-i)(2i+j)^m t^m. \quad (12)$$

Now, expanding the sum over j and using the Maclaurin series expansion for the sum over m , the asymptotic expansion of the symplectic Penner model becomes

$$\begin{aligned} F(t, N, 2) = & \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} N t^{2m-1} \\ & + \sum_{i=0}^{N-1} (N-1-i) [\log(1+(2i+1)t) + \log(1+(2i+2)t)], \end{aligned} \quad (13)$$

the summation over i can be written as follows

$$F(t, N, 2) = \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} N t^{2m-1} + \sum_{p_{\text{odd}}=1}^{2N-1} \frac{1}{2} (2N-1-p) \log(1+pt) + \sum_{p_{\text{even}}=2}^{2N} \frac{1}{2} (2N-p) \log(1+pt), \quad (14)$$

Combining the similar expressions for both odd and even summations, the restrictions may be lifted and gives $\sum_{p=1}^{2N} \frac{1}{2} (2N-p) \log(1+pt)$, with one remaining restricted term namely, $\sum_{p=\text{odd}}^{2N-1} \log(1+pt)^{-\frac{1}{2}}$. The free energy expression becomes

$$F(t, N, 2) = \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} N t^{2m-1} + \log \prod_{p=1}^{2N} (1+pt)^{\frac{1}{2}(2N-p)} - \log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{\frac{1}{2}}. \quad (15)$$

We claim that the last term in which p is restricted to be odd is responsible for the contributions coming from graphs drawn on non-orientable Riemann surfaces. In order to see this, we point out that the combination of the first and the second terms is half of the Penner model, with the size of the matrix doubled. Using the Stirling formula for the logarithm of $\Gamma(\frac{1}{t})$

$$\log(\Gamma(\frac{1}{t})) = -\frac{1}{t} \log t - \frac{1}{t} + \frac{1}{2} \log t + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} t^{2m-1} + \text{const} \quad (16)$$

the symplectic Penner model finally reads:

$$F(t, N, 2) = \frac{1}{2} \log \left[\left(\frac{\sqrt{2\pi t} (et)^{-(t^{-1})}}{\Gamma(\frac{1}{t})} \right)^{2N} \prod_{p=1}^{2N} (1+pt)^{(2N-p)} \right] - \frac{1}{2} \log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt). \quad (17)$$

Writing the first term in equation (17) as $\frac{1}{2} F(t, 2N, 1)$, Then the symplectic Penner model reads:

$$F(t, N, 2) = \frac{1}{2} F(t, 2N, 1) - \frac{1}{2} \log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt). \quad (18)$$

In order to see that the second term is the generating function of the orbifold Euler characteristic $\chi^r(\mathfrak{M}_{g,n})$, we first expand the latter,

$$\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m} t^m \sum_{p_{\text{odd}}=1}^{2N-1} p^m. \quad (19)$$

Now, the restricted sum in the above equation may be lifted using

$$\sum_{p_{\text{odd}}=1}^{2N-1} p^m = \sum_{p=1}^{2N} p^m - \sum_{p=1}^N (2p)^m, \quad (20)$$

and from the power sum formula

$$\sum_{p=1}^N p^m = \frac{N^{m+1}}{m+1} + \frac{N_m}{2} + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k-1} \frac{B_{2k}}{2k} N^{m+1-2k}, \quad (21)$$

equation (19) can be written as,

$$\begin{aligned} \log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt) &= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m} \frac{N^{m+1}}{m+1} (2t)^m \\ &\quad - \sum_{m=1}^{\infty} \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{m} \binom{m}{2k-1} \frac{B_{2k}}{2k} N^{m+1-2k} (2^{2k-1} - 1) (2t)^m. \end{aligned} \quad (22)$$

Combining the above two sums we get

$$\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt) = \sum_{m=1}^{\infty} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(m-1)!(2^{2k-1} - 1)}{(2k)!(m+1-2k)!} B_{2k} (2N)^{m+1-2k} (-t)^m, \quad (23)$$

replacing m by $2k+n-1$ and k by q in the above equation then,

$$\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{-\frac{1}{2}} = -\frac{1}{2} \sum_{\substack{q \geq 0, n > 0 \\ 1-2q-n < 0}} \frac{(2q+n-2)!(2^{2q-1} - 1)}{(2q)! n!} B_{2q} (2N)^n (-t)^{2q+n-1}. \quad (24)$$

Therefore, this shows that $\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{\frac{1}{2}}$ is indeed the generating function of the orbifold Euler characteristic of the moduli space of smooth real algebraic curves of genus $2q$ with n marked points. From Mulase and Waldron [5], the generating function of the orthogonal Penner model $F(2t, 2N, 2)$, is given by the following expression

$$\begin{aligned} F(2t, 2N, 2) &= \frac{1}{2} \sum_{\substack{g \geq 0, n > 0 \\ 2-2g-n < 0}} \frac{(2g+n-3)!(2g-1)}{(2g)! n!} B_{2g} (2N)^n (-t)^{2g+n-2} \\ &\quad + \frac{1}{2} \sum_{\substack{q \geq 0, n > 0 \\ 1-2q-n < 0}} \frac{(2q+n-2)!(2^{2q-1} - 1)}{(2q)! n!} B_{2q} (2N)^n (-t)^{2q+n-1}. \end{aligned} \quad (25)$$

Therefore, we conclude that the generating function for the non-orientable surfaces contributions to the orthogonal Penner model is

$$\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{\frac{1}{2}}. \quad (26)$$

Note that, the structure of the symplectic and the orthogonal Penner matrix models equations (9), (25), is reminiscent of the $SO(N)$ and $Sp(N)$ Chern-Simons gauge theory on S^3 [10].

3 The Continuum Limit of Symplectic-Orthogonal Penner Matrix Models

It is clear from our previous section on the generating functions of the symplectic Penner model, that the continuum limit is obtained by adding half the continuum limit of the ordinary Penner model $F(t, N, 1)$, to the continuum limit of the non orientable surfaces contributions. The continuum limit of the ordinary Penner model was obtained in [7] and [8], where they found that the Penner free energy is the generating function of the orbifold Euler characteristic without punctures

$\chi^c(\mathfrak{M}_{g,0})$. This means that graphs with n punctures are not present in the continuum limit of the free energy. The continuum limit of the free energy of the ordinary Penner model $F(t, N, 1)$ is given by

$$F(\mu) = \frac{1}{2}\mu^2 \log \mu - \frac{1}{12} \log \mu + \sum_{g \geq 2} \frac{1}{(2g-2)} \frac{B_{2g}}{2g} \mu^{2-2g}, \quad (27)$$

where $\frac{1}{2}\mu^2 \log \mu$ and $\frac{1}{12} \log \mu$ are the sphere and the torus contributions to the free energy respectively. While $\frac{B_{2g}}{2g(2g-2)}$ is the orbifold Euler characteristic without punctures $\chi^c(\mathfrak{M}_{g,0})$. Therefore, in order to obtain the continuum limit for the symplectic Penner model one need only focus on the non-orientable contributions part. First, we rewrite equation (19) as follows;

$$\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{-\frac{1}{2}} = -\frac{1}{2} \left[\sum_{p=1}^{2N} \log(1+pt) - \sum_{p=1}^N \log(1+2pt) \right], \quad (28)$$

using the Euler Maclaurin formula

$$\begin{aligned} \sum_{p=1}^{2N} \log(1+pt) &= \frac{1}{2} [f(2N) - f(1)] + \int_1^{2N} \log(1+xt) dx \\ &+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} [f^{(2k-1)}(1) - f^{(2k-1)}(2N)], \end{aligned} \quad (29)$$

where f stands for $\log(1+xt)$, and $f^{(k)}$ is the k th derivative of f , we obtain

$$\begin{aligned} \log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{-\frac{1}{2}} &= \frac{1}{4} \log \frac{1+t}{1+2t} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} (2^{2k-1} - 1) \left(\frac{t}{1+2Nt} \right)^{2k-1} \\ &+ \frac{1}{2} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} \left[\left(\frac{t}{1+t} \right)^{2k-1} - \left(\frac{2t}{1+2t} \right)^{2k-1} \right] - \frac{1}{2} [(1+2t) \log(1+2t) \\ &\frac{1}{2t} (1+2Nt) \log(1+2Nt) - \frac{1}{t} (1+t) \log(1+t) - N]. \end{aligned} \quad (30)$$

The continuum limit of the non-orientable contributions in equation (30) is obtained by first making the natural scaling $t \rightarrow -t/2N$, then set $\mu = 2N(1-t)$, and let $N \rightarrow \infty$, $t \rightarrow 1^3$ such that μ is kept fixed (double scaling). Doing so, we get:

$$F^{NO}(\mu) = \frac{1}{4} \mu \log \mu - \sum_{k \geq 1} \frac{(2^{2k-1} - 1)}{(2k-1)} \frac{B_{2k}}{4k} \mu^{1-2k}, \quad (31)$$

where $F^{NO}(\mu)$ is the non-orientable surfaces contributions to the continuum limit of the symplectic Penner free energy. The term $\frac{1}{4} \mu \log \mu$ is the $g = 0$ contribution and $\frac{(2^{2k-1}-1)}{(2k-1)} \frac{B_{2k}}{4k}$ is the orbifold Euler characteristic of the moduli space of smooth real algebraic curves without marked points $\chi^r(\mathfrak{M}_{g,0})$.

³This is the same as in the ordinary Penner model [8]

Alternatively, the continuum limit may be obtained by summing over all punctures using equation (24). The $g = 0$ contribution corresponds to the following sum

$$\frac{1}{4} \sum_{n>1} \frac{1}{n(n-1)} (2N)^n (-t)^{n-1}. \quad (32)$$

Making the substitution $t \rightarrow \frac{-t}{2N}$, then the above sum reads

$$\frac{N}{2} \sum_{n>1} \frac{1}{n(n-1)} (t)^{n-1} = \frac{N}{2} \left[\sum_{n=2}^{\infty} \frac{(t)^{n-1}}{n-1} - \sum_{n=2}^{\infty} \frac{(t)^{n-1}}{n} \right] \quad (33)$$

which is equivalent to

$$\frac{N}{2} \left[1 + \left(\frac{1-t}{t} \right) \log(1-t) \right]. \quad (34)$$

Then, taking the continuum limit, one can reproduce the $g = 0$ contribution. Similarly, for the higher genus $g \geq 1$, upon using the identity

$$\frac{d^n}{dt^n} (1-t)^{-2g} = \frac{(2g+n-1)!}{(2g-1)!} (1-t)^{-2g-n},$$

the sum over punctures gives

$$- \sum_{q \geq 1} \frac{(2^{2q-1} - 1)}{(2q-1)} \frac{B_{2q}}{4q} \left(\frac{2N(1-t)}{t} \right)^{1-2q}, \quad (35)$$

which in turn implies the continuum limit given by equation (31).

Finally, the free energy of the symplectic Penner model in the continuum limit reads;

$$F(\mu) = \frac{1}{4} \mu^2 \log \mu - \frac{1}{4} \mu \log \mu + \frac{1}{24} \log \mu + \frac{1}{2} \sum_{g \geq 2} \frac{1}{(2g-2)} \frac{B_{2g}}{2g} \mu^{2-2g} - \sum_{k \geq 1} \frac{(2^{2k-1} - 1)}{(2k-1)} \frac{B_{2k}}{4k} \mu^{1-2k}. \quad (36)$$

Therefore, this free energy is related to the orbifold Euler characteristic without punctures. However, differentiating $F(\mu)$ with respect to μ , n -times provided that $n \geq 3$, brings back the punctures on the Riemann surfaces, i.e. we obtain equation (9). Similarly, the continuum limit of the orthogonal Penner model can be obtained using equation (25). Alternatively, this would be equivalent to change the sign for the non-orientable contributions given by equation (36).

It is interesting to note the connection between the non-orientable contributions and $D = 1$ string theory [11]. This connection can be easily seen if we make the Wick rotation $\mu \rightarrow i\mu$ in the density of states $\pi\rho(\mu_F)$ given by the following expression

$$\pi\rho(\mu_F) = \frac{1}{2} \left[-\ln \mu + \sum_{m=1}^{\infty} (2^{2m-1} - 1) \frac{B_{2m}}{m} \frac{1}{\mu^{2m}} \right],$$

integrating the Wick rotated density of states with respect to μ , then we obtain the non-orientable contributions for the free energy of the symplectic Penner model given by (31).

In summary, the generating function formula for the orbifold Euler characteristic, when g is even, is identified and given by the simple formula $\log \prod_{p_{\text{odd}}=1}^{2N-1} (1+pt)^{\frac{1}{2}}$. Moreover, the continuum limits of the symplectic-orthogonal Penner models are obtained and both share the same critical point $t = 1$ with the Penner model [8]. The second author found recently that the generating function for $\chi^c(\mathfrak{M}_{g,n})$ is always present [12] in all generating functions for the virtual Euler characteristics of moduli spaces [2], as yet unidentified, meaning that the orientability is present in all these moduli spaces.

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